Density of States in Diverse Geometries: Verifying Weyl's Theorem for Sphere, Cylinder, Torus, and Cone

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Abstract

The density of states (DoS) quantifies the number of wave modes per wavenumber interval, important in quantum mechanics, acoustics, and photonics. Weyl's theorem predicts a universal DoS, $dN = \frac{Vk^2}{2\pi^2} dk$ (for 3D systems in the short-wavelength limit), that is independent of shape. Following Lambert's 1968 methodology, we derive the DoS for a sphere and cylinder using spherical and cylindrical Bessel functions, respectively, confirming Weyl's result. Extending this, we analyze a torus and cone, employing toroidal approximations and conical coordinates with Legendre functions. All shapes give the same DoS, $dN = \frac{Vk^2}{2\pi^2} dk$, even though they look different, proving shape doesn't matter for tiny waves. This shows Weyl's theorem is very reliable. The results help count wave patterns in tricky shapes and apply to quantum physics, heat in materials, and light in lasers. Our work makes math easy to understand and connects it to real-world uses for different shapes.

Keywords

Density of States, Weyl's Theorem, Sphere, Cylinder, Torus, Cone.